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Skin friction and surface temperature of an insulated flat plate fixed in a fluctuating stream

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The time-mean skin friction of the laminar boundary layer on a flat plate which is fixed at zero incidence in a fluctuating stream is investigated analytically. Flow oscillation amplitude outside the boundary layer is assumed constant along the surface. First, the small velocity-amplitude case is treated, and approximate formulae are obtained in the extreme cases when the frequency is low and high. Next, the finite velocity-amplitude case is treated under the condition of high frequency, and it is found that the formula obtained for the small-amplitude and high-frequency case is also valid. These results show that the increase of the mean skin friction reduces with frequency and is ultimately inversely proportional to the square of frequency.

The corresponding energy equation is also studied simultaneously under the condition of zero heat transfer between the fluid and the surface. It is confirmed that the time-mean surface temperature increases with frequency and tends to be proportional to the square root of frequency. Moreover, it is shown that the time-mean recovery factor can be several times as large as that without flow oscillation.

1. Introduction

Boundary-layer flow problems in oscillatory motions with steady oncoming stream arise in connexion with many interesting and important fluid-mechanical effects. Studies on the fluctuating component of the periodic boundary layer have been made by many authors, for example, Moore (1951), Lighthill (1954) Stuart (1955), Lin (1957), Illingworth (1958). However, few works have been published on the time-mean problems of the periodic boundary layer. Solutions which are restricted to low-frequency oscillation have been obtained for flat-plate flow by Moore & Ostrach (1956) and Kestin, Maeder & Wang (1961). The time-mean characters, including high-frequency oscillation, of the periodic boundary layer near a two-dimensional stagnation point have been studied by Ishigaki (1970).

Among the above-mentioned works, Lin has qualitatively suggested that, in the effect of high-frequency oscillation on time-mean flow, oscillation amplitude variation along a body surface, dU_1/dx, plays an important role and even a large-amplitude oscillation will produce small changes in time-mean flow field if $dU_1/dx = 0$; here U_1 is an external flow oscillation amplitude and x is a distance along a body surface. Ishigaki has shown that time-mean skin friction near a two-dimensional stagnation point is inversely proportional to a half power of frequency for high-frequency oscillation.

Even though time-mean deviation from a steady state may be small when $dU_1/dx = 0$, it will be non-zero because it, of course, arises from non-linear terms of the boundary-layer equation. The main purpose of the present paper is to investigate the effect of flow oscillation, especially of high frequency, on time-mean skin friction of a flat plate under the condition $dU_1/dx = 0$. The results will be contrasted with the former stagnation-point flow results.

With viscous dissipation of kinetic energy taken into account, the corresponding energy equation is also treated simultaneously under the condition of insulated surface. Concerning the oscillating thermometer problem, a few works have been published. Stuart (1955) has treated exactly the effect of flow oscillation on the temperature field on an infinite insulated plate with uniform suction. Maslen & Ostrach (1957) have studied the temperature field on an insulated plate oscillating in a fluid at rest. These two results show that the time-mean surface temperature rises with frequency. It may be of some value to confirm these results for a more complicated case when an infinite insulated plate without suction is fixed in the fluctuating oncoming stream.

2. Small velocity-amplitude case

Let us consider the two-dimensional boundary layer on an insulated flat plate fixed at zero incidence in an unsteady stream of an incompressible fluid. Let x denote distance along the surface from the leading edge, y the normal distance from the surface, and u, v the corresponding velocity components, T the temperature, t time, ν kinematic viscosity, κ thermal diffusivity, c specific heat, and $U(x,t)$ the external flow velocity.

The boundary-layer equations for velocity and temperature, including viscous dissipation, are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}+\frac{\nu}{c}\left(\frac{\partial u}{\partial y}\right)^2, \tag{3}$$

$$u = v = \partial T/\partial y = 0 \quad \text{at} \quad y = 0; \qquad u = U(x,t), \qquad T = T_\infty \quad \text{as} \quad y \to \infty.$$

We shall confine our attention to the external velocity $U(x,t)$ of the form

$$U = U_\infty(1+\epsilon\, e^{i\omega t}), \tag{4}$$

where U_∞, ϵ are constant and ω is frequency.

From the continuity equation (1) we can define a function ψ by

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x. \tag{5}$$

When ϵ is smaller than unity, we may develop the functions ψ and T in the forms

$$\left.\begin{aligned} \psi(x,y,t) &= \psi_0(x,y)+\epsilon\psi_1(x,y)\,e^{i\omega t}+\epsilon^2\{\psi_s(x,y)+\psi_2(x,y)\,e^{2i\omega t}\}+O(\epsilon^3),\\ T(x,y,t) &= T_0(x,y)+\epsilon T_1(x,y)\,e^{i\omega t}+\epsilon^2\{T_s(x,y)+T_2(x,y)\,e^{2i\omega t}\}+O(\epsilon^3), \end{aligned}\right\} \tag{6}$$

where only the real part is to be taken. Substituting the expressions (4), (5), (6) into (2), (3) and equating the same order of ϵ, sets of equations are obtained.

Equations for ψ_0 and T_0 are the steady-state equations and the solutions are the following well-known functions:

$$\psi_0 = (2\nu U_\infty x)^{\frac{1}{2}} f(\eta), \quad T_0 = T_\infty + (U_\infty^2/2c)\, r(\eta); \quad \eta = (U_\infty/2\nu x)^{\frac{1}{2}} y. \tag{7}$$

Equations for ψ_1 and T_1 are

$$i\omega \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_0}{\partial y}\frac{\partial^2 \psi_1}{\partial x \partial y} + \frac{\partial \psi_1}{\partial y}\frac{\partial^2 \psi_0}{\partial x \partial y} - \frac{\partial \psi_0}{\partial x}\frac{\partial^2 \psi_1}{\partial y^2} - \frac{\partial \psi_1}{\partial x}\frac{\partial^2 \psi_0}{\partial y^2} = i\omega U_\infty + \nu \frac{\partial^3 \psi_1}{\partial y^3}, \tag{8a}$$

$$i\omega T_1 + \frac{\partial \psi_0}{\partial y}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial y}\frac{\partial T_0}{\partial x} - \frac{\partial \psi_0}{\partial x}\frac{\partial T_1}{\partial y} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_0}{\partial y} = \kappa \frac{\partial^2 T_1}{\partial y^2} + \frac{2\nu}{c}\frac{\partial^2 \psi_0}{\partial y^2}\frac{\partial^2 \psi_1}{\partial y^2}, \tag{8b}$$

$$\psi_1 = \partial \psi_1/\partial y = \partial T_1/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \partial \psi_1/\partial y = U_\infty, \quad T_1 = 0 \quad \text{as} \quad y \to \infty.$$

Approximate solutions have been obtained in the extreme cases when the frequency parameter $\sigma = \omega x/U_\infty$ is small and large. For small σ Moore's results can be particularized as

$$\psi_1 = (2\nu U_\infty x)^{\frac{1}{2}} \sum_{n=0} (i\sigma)^n g_n(\eta), \quad T_1 = (U_\infty^2/2c) \sum_{n=0} (i\sigma)^n p_n(\eta). \tag{9}$$

For large σ the method due to Illingworth, who has treated the heat-transfer problem under the condition of negligible viscous dissipation, may be appropriate. If

$$\alpha = (2i\sigma)^{-\frac{1}{2}} \quad \text{and} \quad \beta = (i\omega/\nu)^{\frac{1}{2}} y,$$

it may be written as

$$\psi_1 = U_\infty(\nu/i\omega)^{\frac{1}{2}} \sum_{n=0} \alpha^n g_n(\beta), \quad T_1 = (U_\infty^2/2c) \sum_{n=0} \alpha^n p_n(\beta). \tag{10a}$$

Provided that α is small and β is not too large, the following approximations are made when $\eta = \alpha\beta$ is considered:

$$f(\alpha\beta) = \tfrac{1}{2}\alpha^2\beta^2 f''(0) + O(\alpha^5), \quad r(\alpha\beta) = r(0) + \tfrac{1}{2}\alpha^2\beta^2 r''(0) + O(\alpha^5), \tag{11}$$

where the primes denote differentiation with respect to η. Solutions which satisfy the boundary conditions at $\beta = 0$ are

$$\left.\begin{aligned} &g_0 = \beta - 1 + e^{-\beta}, \quad g_1 = g_2 = 0, \\ &g_3 = \tfrac{1}{8}\{4\beta^2 - 13 + (13 + 13\beta + 5\beta^2 + \tfrac{2}{3}\beta^3)\, e^{-\beta}\} f''(0), \\ &p_0 = 0, \quad p_1 = \frac{4\mathscr{P}^{\frac{1}{2}}}{1-\mathscr{P}}\{e^{-\mathscr{P}^{\frac{1}{2}}\beta} - \mathscr{P}^{\frac{1}{2}} e^{-\beta}\} f''(0), \quad p_2 = p_3 = 0, \end{aligned}\right\} \tag{10b}$$

in which $\mathscr{P} = \nu/\kappa$ is the Prandtl number. For the amplitude and phase angle of fluctuating skin friction of order ϵ, the reader may refer to the results of Illingworth. For fluctuating surface temperature expressions are obtained from (9), (10) as

$$\frac{(T_1)_{y=0}}{(T_0)_{y=0} - T_\infty} = 2{\cdot}0 - 2{\cdot}852 i\sigma + 3{\cdot}266 (i\sigma)^2 + O(\sigma^3) \quad (\text{small } \sigma), \tag{12a}$$

$$= 0{\cdot}7193 (i\sigma)^{-\frac{1}{2}} + O(\sigma^{-2}) \quad (\text{large } \sigma). \tag{12b}$$

The amplitude and phase angle of fluctuating surface temperature of order ϵ are shown in figure 1.

Equations for time-independent functions ψ_s and T_s are

$$\nu\frac{\partial^3\psi_s}{\partial y^3}-\frac{\partial\psi_0}{\partial y}\frac{\partial^2\psi_s}{\partial x\,\partial y}-\frac{\partial\psi_s}{\partial y}\frac{\partial^2\psi_0}{\partial x\,\partial y}+\frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_s}{\partial y^2}+\frac{\partial\psi_s}{\partial x}\frac{\partial^2\psi_0}{\partial y^2}$$
$$=\frac{1}{2}\left\{\frac{\partial\psi_{1r}}{\partial y}\frac{\partial^2\psi_{1r}}{\partial x\,\partial y}+\frac{\partial\psi_{1i}}{\partial y}\frac{\partial^2\psi_{1i}}{\partial x\,\partial y}-\frac{\partial\psi_{1r}}{\partial x}\frac{\partial^2\psi_{1r}}{\partial y^2}-\frac{\partial\psi_{1i}}{\partial x}\frac{\partial^2\psi_{1i}}{\partial y^2}\right\}, \tag{13a}$$

$$\kappa\frac{\partial^2 T_s}{\partial y^2}-\frac{\partial\psi_0}{\partial y}\frac{\partial T_s}{\partial x}-\frac{\partial\psi_s}{\partial y}\frac{\partial T_0}{\partial x}+\frac{\partial\psi_0}{\partial x}\frac{\partial T_s}{\partial y}+\frac{\partial\psi_s}{\partial x}\frac{\partial T_0}{\partial y}$$
$$=\frac{1}{2}\left\{\frac{\partial\psi_{1r}}{\partial y}\frac{\partial T_{1r}}{\partial x}+\frac{\partial\psi_{1i}}{\partial y}\frac{\partial T_{1i}}{\partial x}-\frac{\partial\psi_{1r}}{\partial x}\frac{\partial T_{1r}}{\partial y}-\frac{\partial\psi_{1i}}{\partial x}\frac{\partial T_{1i}}{\partial y}\right\}$$
$$-\frac{\nu}{c}\left[2\frac{\partial^2\psi_0}{\partial y^2}\frac{\partial^2\psi_s}{\partial y^2}+\frac{1}{2}\left\{\left(\frac{\partial^2\psi_{1r}}{\partial y^2}\right)^2+\left(\frac{\partial\psi_{1i}}{\partial y^2}\right)^2\right\}\right], \tag{13b}$$

$$\psi_s=\partial\psi_s/\partial y=\partial T_s/\partial y=0 \quad \text{at} \quad y=0; \qquad \partial\psi_s/\partial y=T_s=0 \quad \text{as} \quad y\to\infty,$$

where subscripts r and i respectively denote real and imaginary part of function.

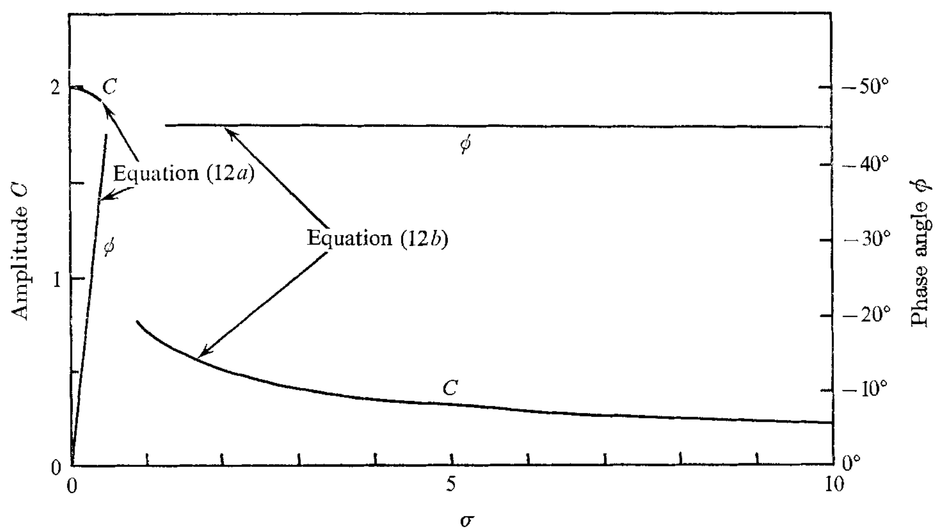

FIGURE 1. Amplitude and phase angle of fluctuating surface temperature.

When σ is small, appropriate forms may be written as

$$\psi_s=(2\nu U_\infty x)^{\frac{1}{2}}\sum_{n=0}\sigma^n G_n(\eta), \quad T_s=(U_\infty^2/2c)\sum_{n=0}\sigma^n P_n(\eta). \tag{14}$$

Substituting (9), (14) into (13) and equating the same order of σ, we have from (13a)

$$\left.\begin{aligned}
&G_0'''+fG_0''\qquad\quad +f''G_0=-\tfrac{1}{2}g_0g_0'',\\
&G_1'''+fG_1''-2f'G_1'+3f''G_1=0,\\
&G_2'''+fG_2''-4f'G_2'+5f''G_2=g_1'^2-2g_0'g_2'+\tfrac{5}{2}g_0''g_2-\tfrac{3}{2}g_1g_1''+\tfrac{1}{2}g_0g_2'',\\
&\qquad\vdots\qquad\qquad\qquad\qquad\qquad\vdots\\
&G_0=G_0'=G_1=G_1'=G_2=G_2'=\ldots=0 \quad \text{at} \quad \eta=0;\\
&G_0'=G_1'=G_2'=\ldots=0 \quad \text{as} \quad \eta\to\infty,
\end{aligned}\right\} \tag{15a}$$

and from (13*b*)

$$\left.\begin{aligned}
&\mathscr{P}^{-1}P_0''+fP_0' = -r'G_0-\tfrac{1}{2}g_0p_0'-4f''G_0''-(g_0'')^2,\\
&\mathscr{P}^{-1}P_1''+fP_1'-2f'P_1 = -3r'G_1-4f''G_1'',\\
&\mathscr{P}^{-1}P_2''+fP_2'-4f'P_2 = -5r'G_2+\tfrac{5}{2}g_2p_0'+g_1'p_1-\tfrac{3}{2}g_1p_1'-2g_0'p_2+\tfrac{1}{2}g_0p_2'\\
&\qquad\qquad -4f''G_2''+2g_0''g_2''-(g_1'')^2,\\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots\\
&P_0'=P_1'=P_2'=\ldots=0 \quad \text{at} \quad \eta=0;\\
&P_0=P_1=P_2=\ldots=0 \quad \text{as} \quad \eta\to\infty.
\end{aligned}\right\} \tag{15b}$$

Here G_0 and P_0 are the quasi-steady state solutions, and we have

$$G_0=\tfrac{1}{16}(-f+\eta f'+\eta^2f''),\quad P_0=\tfrac{1}{2}r+\tfrac{7}{16}\eta r'+\tfrac{1}{16}\eta^2r''.$$

Functions G_1 and P_1 are identically zero and G_2, P_2 are obtained numerically for $\mathscr{P}=0{\cdot}72$.

When σ is large, substitution of (10*a*) and (10*b*) into the right-hand side of (13*a*) yields

$$\nu\frac{\partial^3\psi_s}{\partial y^3}-\frac{\partial\psi_0}{\partial y}\frac{\partial^2\psi_s}{\partial x\,\partial y}-\frac{\partial\psi_s}{\partial y}\frac{\partial^2\psi_0}{\partial x\,\partial y}+\frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_s}{\partial y^2}+\frac{\partial\psi_s}{\partial x}\frac{\partial^2\psi_0}{\partial y^2}$$

$$\begin{aligned}&=\frac{3}{128}\frac{U_\infty^2}{x\sigma^{\frac{3}{2}}}\{(13-2z^2+\tfrac{8}{3}z^3)\cos z\,e^{-z}+(13+10z+2z^2)\sin z\,e^{-z}\\ &\qquad-(13+26z+16z^2+\tfrac{8}{3}z^3)\,e^{-2z}\}f''(0)+O(\sigma^{-3}),\end{aligned}\tag{16}$$

in which $z=(\omega/2\nu)^{\frac{1}{2}}y$. The particular solution, ψ_{sp}, may be written as

$$\psi_{sp}=(U_\infty/\sigma^{\frac{5}{2}})(\nu/2\omega)^{\frac{1}{2}}\{G_c(x,z)\cos z\,e^{-z}+G_d(x,z)\sin z\,e^{-z}+G_e(x,z)\,e^{-2z}\}. \tag{17a}$$

The equation for G_e, for example, is

$$\begin{aligned}&\frac{\partial^3G_e}{\partial z^3}-6\frac{\partial^2G_e}{\partial z^2}+12\frac{\partial G_e}{\partial z}-8G_e+\left(\frac{1}{\sigma^{\frac{3}{2}}}\right)\left\{2x\frac{\partial G_e}{\partial x}-5G_e\right\}f''(0)\\
&-\left(\frac{z}{\sigma^{\frac{3}{2}}}\right)\left\{2x\frac{\partial^2G_e}{\partial x\,\partial z}-4x\frac{\partial G_e}{\partial x}-6\frac{\partial G_e}{\partial z}+12G_e\right\}f''(0)-\left(\frac{z^2}{\sigma^{\frac{3}{2}}}\right)\\
&\times\left\{\frac{1}{2}\frac{\partial^2G_e}{\partial z^2}-2\frac{\partial G_e}{\partial z}+2G_e\right\}f''(0)=-\tfrac{3}{32}(13+26z+16z^2+\tfrac{8}{3}z^3)f''(0)+O(\sigma^{-\frac{3}{2}}).\end{aligned}$$

Therefore expanding $\qquad G_e(x,z)=\sum\limits_{n=0}\sigma^{-\frac{1}{2}n}G_{en}(z),$

we obtain $\quad G_{e0}=\tfrac{1}{821}(180+147z+42z^2+4z^3)f''(0),\quad G_{e1}=G_{e2}=0.$

Similarly we have

$$G_c(x,z)=\tfrac{1}{64}(39z+30z^2+4z^3)f''(0)+O(\sigma^{-\frac{3}{2}}),$$

$$G_d(x,z)=\tfrac{1}{64}(-39z+4z^3)f''(0)+O(\sigma^{-\frac{3}{2}}).$$

It is readily understood that boundary conditions at the wall are to be adjusted by solving the homogeneous equation under the appropriate boundary conditions which are estimated from the particular solutions. This homogeneous solution ψ_{sh} may be expressed as

$$\psi_{sh} = (2\nu U_\infty x)^{\frac{1}{2}}\sigma^{-\frac{5}{4}}\sum_{n=0}\sigma^{-\frac{1}{2}n}G_{0n}(\eta). \tag{17b}$$

Therefore the contribution to the time-mean skin friction is of the order of $\sigma^{-\frac{5}{4}}$ and will be neglected in the after skin friction estimation.

As to the time-mean energy equation (13*b*), substitution of (10) and (17) yields

$$\kappa\frac{\partial^2 T_s}{\partial y^2}-\frac{\partial\psi_0}{\partial y}\frac{\partial T_s}{\partial x}+\frac{\partial\psi_0}{\partial x}\frac{\partial T_s}{\partial y} = -\frac{U_\infty^3}{2cx}\sigma e^{-2z}+O(\sigma^{-\frac{1}{2}}). \tag{18}$$

The particular solution T_{sp} may be expressed as

$$T_{sp} = (U_\infty^2/2c)\,P_e(x,z)\,e^{-2z} \tag{19a}$$

and we have
$$P_e = -\tfrac{1}{2}\mathscr{P}+O(\sigma^{-\frac{3}{2}}).$$

Then the homogeneous solution T_{sh} may be expressed as

$$T_{sh} = (U_\infty^2/2c)\,\sigma^{\frac{1}{2}}\sum_{n=0}\sigma^{-\frac{1}{2}n}P_n(\eta), \tag{19b}$$

and satisfy the following equations and boundary conditions:

$$\mathscr{P}^{-1}P_i''+fP_i'+(i-1)f'P_i = 0 \quad (i = 0,1,2),$$

$$P_0' = -\mathscr{P},\quad P_1 = P_2 = 0 \quad \text{at} \quad \eta = 0; \qquad P_0 = P_1 = P_2 = 0 \quad \text{as} \quad \eta\to\infty.$$

Functions P_1, P_2 are identically zero and P_0 is found numerically for $\mathscr{P} = 0{\cdot}72$.

Now we can obtain the approximate expressions of time-mean skin friction and surface temperature. If we write the time-mean skin friction,

$$\bar\tau_w = \frac{1}{2\pi}\int_0^{2\pi}\mu\,(\partial u/\partial y)_{y=0}\,dt,$$

as
$$\bar\tau_w/\tau_{w0} = 1+\epsilon^2 A(\sigma)+O(\epsilon^4), \tag{20}$$

in which τ_{w0} is the skin friction without oscillation and μ is viscosity, then

$$A(\sigma) = \tfrac{3}{16}-0{\cdot}621\sigma^2+O(\sigma^4) \quad (\text{small } \sigma), \tag{20a}$$

$$= 0{\cdot}1875\sigma^{-2}+O(\sigma^{-\frac{5}{2}}) \quad (\text{large } \sigma). \tag{20b}$$

The function $A(\sigma)$ is shown in figure 2. We can see that $A(\sigma)$ decreases with σ abruptly and tends to zero for a medium value of σ. As compared with this, the corresponding function for stagnation-point flow decreases with frequency very gradually.

For the time-mean surface temperature

$$\bar T_w = \frac{1}{2\pi}\int_0^{2\pi}(T)_{y=0}\,dt$$

we write
$$\frac{\bar T_w - T_\infty}{(T_0)_{y=0}-T_\infty} = 1+\epsilon^2 B(\sigma)+O(\sigma^4). \tag{21}$$

It follows
$$B(\sigma) = \tfrac{1}{2}+2{\cdot}448\sigma^2+O(\sigma^4) \quad (\text{small } \sigma), \tag{21a}$$

$$= 1{\cdot}4568\sigma^{\frac{1}{2}}-0{\cdot}425+O(\sigma^{-1}) \quad (\text{large } \sigma). \tag{21b}$$

The function $B(\sigma)$ is shown in figure 3. The asymptotic value for very large σ, first term only in (21*b*), is also shown by broken line.

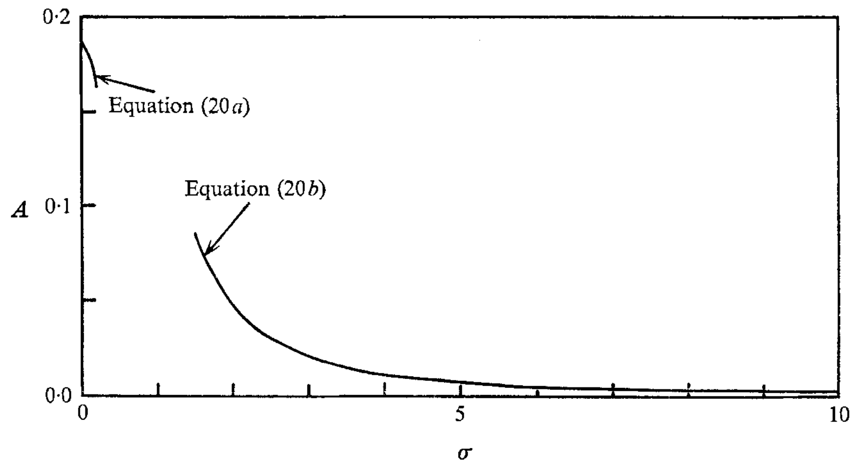

FIGURE 2. Plot of A with frequency parameter σ.

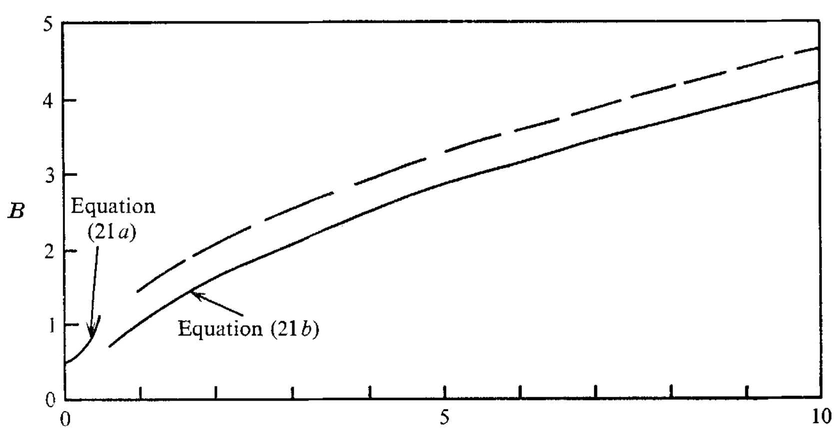

FIGURE 3. Plot of B with frequency parameter σ.

3. Finite velocity-amplitude case

The basic equations in the boundary layer are identical with those of preceding section, that is, (1), (2) and (3). The external flow velocity outside the boundary layer is given by (4) without restriction on ϵ.

Following Lin's method, functions ψ, defined in (5), and T may be expressed as the sum of a time-mean and a time-dependent component as

$$\left.\begin{aligned} \psi(x,y,t) &= \overline{\psi}(x,y)+\psi_t(x,y,t) \quad (\overline{\psi}_t = 0), \\ T(x,y,t) &= \overline{T}(x,y)+T_t(x,y,t) \quad (\overline{T}_t = 0), \end{aligned}\right\} \tag{22}$$

where a bar over the symbols denotes time-mean quantities. Substituting (22) into (1), (2), (3), (5) and taking its time average, time-mean equations are obtained. Subtracting each time-mean equation from the corresponding full equation, time-dependent equations are obtained.

The equation for time-dependent velocity is

$$\nu \frac{\partial^3 \psi_t}{\partial y^3} - \frac{\partial^2 \psi_t}{\partial t\, \partial y} + i\omega\epsilon U_\infty e^{i\omega t} = \xi, \tag{23}$$

$$\psi_t = \partial\psi_t/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \partial\psi_t/\partial y = \epsilon U_\infty e^{i\omega t} \quad \text{as} \quad y \to \infty,$$

where

$$\xi = \frac{\partial\overline{\psi}}{\partial y}\frac{\partial^2\psi_t}{\partial x\,\partial y} + \frac{\partial\psi_t}{\partial y}\frac{\partial^2\overline{\psi}}{\partial x\,\partial y} + \frac{\partial\psi_t}{\partial y}\frac{\partial^2\psi_t}{\partial x\,\partial y} - \overline{\frac{\partial\psi_t}{\partial y}\frac{\partial^2\psi_t}{\partial x\,\partial y}}$$
$$- \left(\frac{\partial\overline{\psi}}{\partial x}\frac{\partial^2\psi_t}{\partial y^2} + \frac{\partial\psi_t}{\partial x}\frac{\partial^2\overline{\psi}}{\partial y^2} + \frac{\partial\psi_t}{\partial x}\frac{\partial^2\psi_t}{\partial y^2} - \overline{\frac{\partial\psi_t}{\partial x}\frac{\partial^2\psi_t}{\partial y^2}}\right).$$

The equation for time-mean velocity is

$$\frac{\partial\overline{\psi}}{\partial y}\frac{\partial^2\overline{\psi}}{\partial x\,\partial y} + \overline{\frac{\partial\psi_t}{\partial y}\frac{\partial^2\psi_t}{\partial x\,\partial y}} - \left(\frac{\partial\overline{\psi}}{\partial x}\frac{\partial^2\overline{\psi}}{\partial y^2} + \overline{\frac{\partial\psi_t}{\partial x}\frac{\partial^2\psi_t}{\partial y^2}}\right) = \nu\frac{\partial^3\overline{\psi}}{\partial y^3}, \tag{24}$$

$$\overline{\psi} = \partial\overline{\psi}/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \partial\overline{\psi}/\partial y = U_\infty \quad \text{as} \quad y \to \infty.$$

In the case of high frequency, it may be expected that the left-hand side of (23) is of major importance and the method of successive approximations is appropriate. We let

$$\psi_t = \psi_{t0} + \psi_{t1}, \quad \overline{\psi} = \overline{\psi}_0 + \overline{\psi}_1. \tag{25}$$

Under the conditions

$$\frac{\partial^2\psi_t}{\partial t\,\partial y} \gg \frac{\partial\psi_t}{\partial y}\frac{\partial^2\overline{\psi}}{\partial x\,\partial y}, \quad \frac{\partial\psi_t}{\partial y}\frac{\partial^2\psi_t}{\partial x\,\partial y} \quad \text{or} \quad \sigma \gg 1, \epsilon, \tag{26}$$

the first approximation to the time-dependent component is obtained from (23) after letting ξ equal zero. The solution has been obtained by Lin as

$$\psi_{t0} = \epsilon U_\infty (\nu/i\omega)^{\frac{1}{2}} \{(i\omega/\nu)^{\frac{1}{2}} y - 1 + \exp(-y(i\omega/\nu)^{\frac{1}{2}})\} e^{i\omega t}, \tag{27}$$

which is independent of x. Substitution of this solution therefore reduces (24) to the equation of the flow without oscillation and the first approximation to the time-mean component $\overline{\psi}_0$ is equal to ψ_0 in (7). Therefore it follows that, even if velocity amplitude becomes large, the effect of high-frequency oscillation on time-mean flow field on a flat plate may be negligible to the first approximation.

Substituting (27) into ξ in (23), we have the second-approximation equation as

$$\nu\frac{\partial^3\psi_{t1}}{\partial y^3} - \frac{\partial^2\psi_{t1}}{\partial t\,\partial y} = \frac{\partial\psi_{t0}}{\partial y}\frac{\partial^2\psi_0}{\partial x\,\partial y} - \frac{\partial\psi_0}{\partial x}\frac{\partial^2\psi_{t0}}{\partial y^2}, \tag{28}$$

$$\psi_{t1} = \partial\psi_{t1}/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \partial\psi_{t1}/\partial y = 0 \quad \text{as} \quad y \to \infty;$$

$$\nu\frac{\partial^3\overline{\psi}_1}{\partial y^3} - \frac{\partial\psi_0}{\partial y}\frac{\partial^2\overline{\psi}_1}{\partial x\,\partial y} - \frac{\partial\overline{\psi}_1}{\partial y}\frac{\partial^2\psi_0}{\partial x\,\partial y} + \frac{\partial\psi_0}{\partial x}\frac{\partial^2\overline{\psi}_1}{\partial y^2} + \frac{\partial\overline{\psi}_1}{\partial x}\frac{\partial^2\psi_0}{\partial y^2} = \overline{\frac{\partial\psi_{t0}}{\partial y}\frac{\partial^2\psi_{t1}}{\partial x\,\partial y}} - \overline{\frac{\partial\psi_{t1}}{\partial x}\frac{\partial^2\psi_{t0}}{\partial y^2}}, \tag{29}$$

$$\overline{\psi}_1 = \partial\overline{\psi}_1/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \partial\overline{\psi}_1/\partial y = 0 \quad \text{as} \quad y \to \infty.$$

When approximation (11) for ψ_0 is also made, a method similar to the small-amplitude case is also applicable. The calculations are somewhat lengthy but straightforward and the details are abridged here. The results so obtained are the same as those of the small-amplitude case and higher harmonics do not yet appear at this stage.

For time-dependent temperature we have

$$\kappa \frac{\partial^2 T_t}{\partial y^2} - \frac{\partial T_t}{\partial t} + \frac{\nu}{c}\left\{2\frac{\partial^2 \overline{\psi}}{\partial y^2}\frac{\partial^2 \psi_t}{\partial y^2} + \left(\frac{\partial^2 \psi_t}{\partial y^2}\right)^2 - \overline{\left(\frac{\partial^2 \psi_t}{\partial y^2}\right)^2}\right\} = \zeta, \tag{30}$$

$$\partial T_t/\partial y = 0 \quad \text{at} \quad y = 0; \qquad T_t = 0 \quad \text{as} \quad y \to \infty,$$

where

$$\zeta = \frac{\partial \overline{\psi}}{\partial y}\frac{\partial T_t}{\partial x} + \frac{\partial \psi_t}{\partial y}\frac{\partial \overline{T}}{\partial x} + \frac{\partial \psi_t}{\partial y}\frac{\partial T_t}{\partial x} - \overline{\frac{\partial \psi_t}{\partial y}\frac{\partial T_t}{\partial x}} - \left(\frac{\partial \overline{\psi}}{\partial x}\frac{\partial T_t}{\partial y} + \frac{\partial \psi_t}{\partial x}\frac{\partial \overline{T}}{\partial y} + \frac{\partial \psi_t}{\partial x}\frac{\partial T_t}{\partial y} - \overline{\frac{\partial \psi_t}{\partial x}\frac{\partial T_t}{\partial y}}\right).$$

Similarly in (23), ζ in (30) is neglected for the first approximation under the conditions

$$\frac{\partial T_t}{\partial t} \gg \frac{\partial \psi_t}{\partial y}\frac{\partial \overline{T}}{\partial x}, \quad \frac{\partial \overline{\psi}}{\partial y}\frac{\partial T_t}{\partial x}, \quad \frac{\partial \psi_t}{\partial y}\frac{\partial T_t}{\partial x} \quad \text{or} \quad \sigma \gg \epsilon\frac{\overline{T}}{T_t}, 1, \epsilon. \tag{31a}$$

Moreover, an assumption is made that

$$\left(\frac{\partial^2 \psi_t}{\partial y^2}\right)^2 \gg \frac{\partial^2 \overline{\psi}}{\partial y^2}\frac{\partial^2 \psi_t}{\partial y^2} \quad \text{or} \quad \sigma \gg \frac{1}{\epsilon^2}, \tag{31b}$$

that is, ϵ is not too small. The simplified equation for T_{t0}, first approximation to the time-dependent temperature, is

$$\kappa \frac{\partial^2 T_{t0}}{\partial y^2} - \frac{\partial T_{t0}}{\partial t} = \frac{\nu}{c}\left\{\overline{\left(\frac{\partial^2 \psi_t}{\partial y^2}\right)^2} - \left(\frac{\partial^2 \psi_t}{\partial y^2}\right)^2\right\}. \tag{32a}$$

The solution is

$$T_{t0} = \frac{\epsilon^2 U_\infty^2}{4c}\frac{\mathscr{P}}{(2-\mathscr{P})}\left\{\left(\frac{2}{\mathscr{P}}\right)^{\frac{1}{2}} \exp\left(-\left(\frac{2i\omega\mathscr{P}}{\nu}\right)^{\frac{1}{2}} y\right) - \exp\left(-2\left(\frac{i\omega}{\nu}\right)^{\frac{1}{2}} y\right)\right\} e^{2i\omega t}. \tag{32b}$$

It can be seen that the second-harmonic fluctuation, which is independent of x, becomes predominant and surface-temperature amplitude is independent of ω.

The time-mean energy equation is

$$\frac{\partial \overline{\psi}}{\partial y}\frac{\partial \overline{T}}{\partial x} - \frac{\partial \overline{\psi}}{\partial x}\frac{\partial \overline{T}}{\partial y} + \left(\overline{\frac{\partial \psi_t}{\partial y}\frac{\partial T_t}{\partial x}} - \overline{\frac{\partial \psi_t}{\partial x}\frac{\partial T_t}{\partial y}}\right) = \kappa\frac{\partial^2 \overline{T}}{\partial y^2} + \frac{\nu}{c}\left\{\left(\frac{\partial^2 \overline{\psi}}{\partial y^2}\right)^2 + \overline{\left(\frac{\partial^2 \psi_t}{\partial y^2}\right)^2}\right\}, \tag{33}$$

$$\partial \overline{T}/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \overline{T} = T_\infty \quad \text{as} \quad y \to \infty.$$

The first approximation of time-mean temperature, $\overline{T}_0$, is obtained from the equation

$$\frac{\partial \psi_0}{\partial y}\frac{\partial \overline{T}_0}{\partial x} - \frac{\partial \psi_0}{\partial x}\frac{\partial \overline{T}_0}{\partial y} = \kappa\frac{\partial^2 \overline{T}_0}{\partial y^2} + \frac{\nu}{c}\left\{\left(\frac{\partial^2 \psi_0}{\partial y^2}\right)^2 + \overline{\left(\frac{\partial^2 \psi_{t0}}{\partial y^2}\right)^2}\right\}. \tag{34}$$

If we let

$$\overline{T}_0 = \overline{T}_{0a} + \overline{T}_{0b}, \tag{35}$$

the equation and boundary conditions for $\overline{T}_{0a}$ can be chosen so as to agree with those without oscillation and the solution is equal to T_0 in (7). Using (27), the equation for $\overline{T}_{0b}$ is therefore given as

$$\kappa\frac{\partial^2 \overline{T}_{0b}}{\partial y^2} - \frac{\partial \psi_0}{\partial y}\frac{\partial \overline{T}_{0b}}{\partial x} + \frac{\partial \psi_0}{\partial x}\frac{\partial \overline{T}_{0b}}{\partial y} = -\frac{\epsilon^2 U_\infty^2}{2c}\,\omega\, e^{-2z}, \tag{36}$$

$$\partial \overline{T}_{0b}/\partial y = 0 \quad \text{at} \quad y = 0; \qquad \overline{T}_{0b} = 0 \quad \text{as} \quad y \to \infty.$$

Approximations (11) are also made and a method similar to (18) leads us to the same results as the small-amplitude case. Therefore the assumption in (31*a*) becomes clear as

$$\sigma \gg \epsilon\overline{T}/T_t \quad \text{or} \quad \sigma \gg \epsilon^2.$$

At the end of this section, it may be helpful to make some discussions about the temperature-field results obtained in these two sections. For large value of σ, Stuart's exact solution for a flat plate with uniform suction can be reduced to

$$(T)_{y=0} = \frac{U_\infty^2}{2c}\left[(1+\epsilon^2(\tfrac{1}{2}\lambda)^{\frac{1}{2}}) + \frac{4\epsilon}{1+\mathscr{P}^{\frac{1}{2}}}\left(\frac{\mathscr{P}}{\lambda}\right)^{\frac{1}{2}}\cos(\omega t - \tfrac{1}{4}\pi) + \frac{\epsilon^2\mathscr{P}^{\frac{1}{2}}}{2(2^{\frac{1}{2}}+\mathscr{P}^{\frac{1}{2}})}\cos 2\omega t\right], \quad (37)$$

in which $\lambda = \omega\nu/v_w^2$ and v_w is constant suction velocity. The present result for a flat plate without suction, deduced from these two sections, is

$$(T)_{y=0} = \frac{U_\infty^2}{2c}\left[\{r(0)+\epsilon^2\sigma^{\frac{1}{2}}P_0(0)\} + \frac{4\epsilon}{1+\mathscr{P}^{\frac{1}{2}}}\left(\frac{\mathscr{P}}{\sigma}\right)^{\frac{1}{2}} f''(0)\cos(\omega t - \tfrac{1}{4}\pi) + \frac{\epsilon^2\mathscr{P}^{\frac{1}{2}}}{2(2^{\frac{1}{2}}+\mathscr{P}^{\frac{1}{2}})}\cos 2\omega t\right]. \quad (38)$$

Comparing these results, the time-mean surface temperatures are of the same character, except that the latter coefficient depends on the Prandtl number, and the only difference in the fluctuating surface temperature is that the latter first-harmonic fluctuation is multiplied by $f''(0)$. Therefore the characters of the temperature field described by Stuart are also valid in this case.

In connexion with the mean surface temperature, we attempt here to obtain the temperature recovery factor for this case of fluctuating flow. If we consider the time-mean stagnation temperature of the external stream, $\overline{T}_{\infty s}$, as

$$\overline{T}_{\infty s} = T_\infty + (U_\infty^2/2c)(1+\tfrac{1}{2}\epsilon^2), \quad (39)$$

the dimensionless mean surface temperature γ is approximately given for high frequency and $\mathscr{P} = 0{\cdot}72$ as

$$\gamma = \frac{\overline{T}_w - \overline{T}_\infty}{\overline{T}_{\infty s} - \overline{T}_\infty} = \frac{0{\cdot}848\{1+\epsilon^2(1{\cdot}457\,\sigma^{\frac{1}{2}} - 0{\cdot}425)\}}{1+\frac{1}{2}\epsilon^2}. \quad (40)$$

This dimensionless temperature is shown in figure 4, and we can see that it can be several times as large as that without flow oscillation ($\epsilon = 0$).

4. Concluding remarks

In order to give the quantitative verification of Lin's theory, approximate velocity solutions are obtained in the case when the flow oscillation amplitude does not change along the surface. The solutions of §§ 2 and 3 show that the time-mean skin friction becomes greater than the steady value and is proportional to the square of the velocity-amplitude ratio ϵ. The contribution of flow oscillation to the mean skin friction decreases with frequency abruptly and is ultimately inversely proportional to the square of the frequency. This result may be contrasted with the former result of stagnation-point flow in which the flow oscillation amplitude changes along the surface. The time-dependent solutions show that the trends are in accord with Lighthill's theory.

The corresponding energy equation is also treated simultaneously under the condition of zero heat transfer between the fluid and the wall. At the end of § 3 comparison with Stuart's result is made, and we can see that two results are in same character at high frequency. Moreover, the dimensionless surface temperature is shown in figure 4, and it can be seen that the time-mean surface temperature can be much greater than the stagnation temperature of the external stream. We know the resonance tube as a device to produce high temperature. The resonance tube is a cylindrical resonator, closed at the downstream

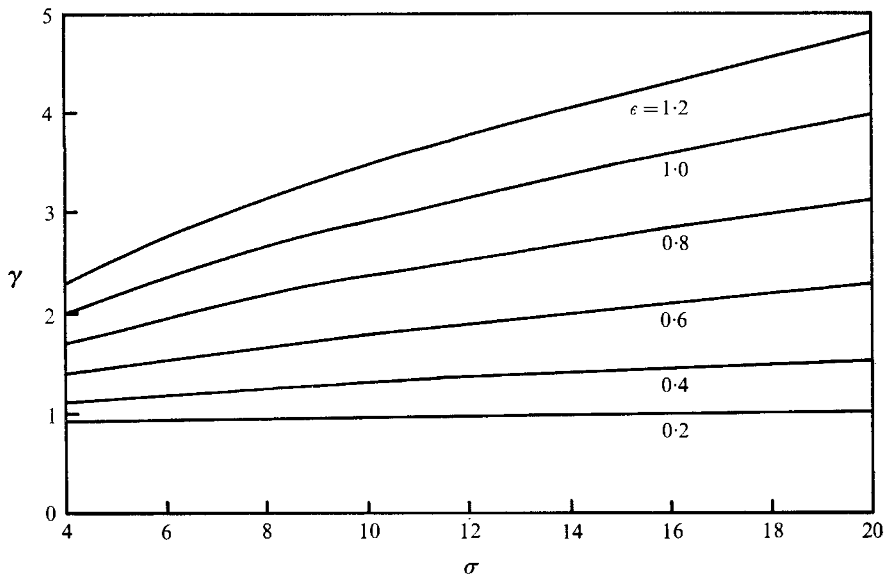

FIGURE 4. Dimensionless mean surface temperature as a function of frequency parameter σ for several values of ϵ.

end, which is excited to oscillation by an air jet. The reason why the tube wall temperature rises higher than the stagnation temperature of the air jet has been hypothesized to be due to shock waves inside the resonance tube. As yet, however, no adequate theory of resonance tube heating has been published and this hypothesis has been questioned by Sibulkin (1963). The effect of viscous dissipation combined with oscillation on the resonance tube heating has not been clarified and may be the subject for a future study.

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